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SEBEXTENSION OF PLURISUBHARMONIC FUNCTIONS AND APPLICATIONS

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Preliminaries

1. Reasons for selecting topic

Extension object of complex analysis: holomorphic and micromorphic mappings, analytic sets, currents, etc, always is one of the problems of complex analysis as well as plurispotential theory. One of the issues most concerned and researched and considered as the center of plurispotential theory is subextension of plurisubharmonic functions. Therefore, as well as mentioned issues, we should put emphasis on examining problems about extension plurisubharmonic functions when researching problems about plurispotential theory. However, because plurisubharmonic functions are defined by inequalities then in plurispotential theory, one consider subextension problem for these functions. In this thesis, we spend most of the content presenting problem of subextension of unbounded plurisubharmonic function class, as well as m- unbounded subharmonic functions. Mentioned issues have recently concerned and researched within the last 10 years .

From 1994 to 2004, Cegrell, one of top world experts about pluritential theory, built up operator Monge - Ampre for some unbounded local plurisubharmonic function classes. He brought out $\mathcal{E}_p(\Omega), \mathcal{F}_p(\Omega), \mathcal{F}(\Omega), \mathcal{N}(\Omega)$ v $\mathcal{E}(\Omega)$. Those are different unbounded plurisubharmonic functions classes in hyperconvex domain $\Omega \subset \mathbb{C}^n$ where operator $(dd^c.)^n$ can be determined and continuous in decreased sequences. In which $\mathcal{E}(\Omega)$ is the largest class where operator Monge - Ampre is defined as a Radon degree. Since then, they started shifting concentration from problems about subextension to these classes.

In 2003, Cegrell and Zeriahi researched problems about subextension for class $\mathcal{F}(\Omega)$ a subunit of class $\mathcal{E}(\Omega)$. the authors proved that: If $\Omega \Subset \widetilde{\Omega}$ are bounded hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$, then $\widetilde{u} \in \mathcal{F}(\Omega)$ exists so that $\widetilde{u} \leq u$ in Ω, $\widetilde{u}$ is later called subextension of u from Ω to $\widetilde{\Omega}$. The important thing is the authors' estimation on operator Monge - Ampre mass of $(dd^c\widetilde{u})^n$ and $(dd^c u)^n$ measures through inequalities $\int\limits_{\widetilde{\Omega}} (dd^c\widetilde{u})^n \leq \int\limits_{\Omega} (dd^c u)^n$. This result can be considered as the first the resultof researching problems about subextension of unbounded plurisubharmonic functions. After that, P. H. Hiep, Benelkourchi continues researching this problem for different function class such as $\mathcal{E}_p(\Omega)$, $\mathcal{E}_\chi(\Omega)$. Examining problems about subextension in Cegrell classes with boundary values by Czyż, Hed in 2008. We will present Czyż and Hed's results further in the beginning of Overview in this thesis. The throughout topic of this thesis is the relationship between $(dd^c\widetilde{u})^n$ and $1_\Omega(dd^c u)^n$ measures with $\widetilde{u}$ subextension of u. Most of the authors Cegrell- Zeriahi's, P.H.Hiep's, Benelkourchi's or Czyż's and Hed's results stop at estimating the relationship between mass total of $(dd^c\widetilde{u})^n$ and mass of $(dd^c u)^n$. So that, researching subextension of plurisubharmonic

functions which can control Monge- Ampre measures of subextension of functions and given functions is an open question. In 2014, L. M. Hi, N. X. Hng researched problems about subextension for class $\mathcal{F}(\Omega, f)$. The important thing is that they proved equation about Monge-Ampère measures of subextension of functions and given functions. Therefore, the problem that needs researching is the extension of results for larger function class, class $\mathcal{E}_\chi(\Omega, f)$?

The next problem which is concerned and researched in this thesis is establishing subextension of plurisubharmonic functions in unbounded domain. We know that defining subextension $\widetilde{u}$ of u needs solving Monge-Ampre equation. However, solving Monge-Ampère equation in unbounded domain in $\mathbb{C}^n$ is not simple. In 2014,an important result in solving Monge-Ampre equation for unbounded hyperconvex domain in $\mathbb{C}^n$ were proposed by L. M. Hai, N. V. Trao, N. X. Hong. That gives direction for us to examine the problem about subextension of plurisubharmonic functions in class $\mathcal{F}(\Omega, f)$ with Ω unbounded hyperconvex domain. As an application of the mentioned result, in the next section of the thesis, we study approximation of plurisubharmonic functions by an increasing sequence of plurisubharmonic functions defined on larger domain.

In chapter 4 of this thesis, we examine subextension for function class m-subharmonic. As we have known, extending plurisubharmonic functions class is studied by some authors such as: Z. Blocki, S. Dinew, S. Kolodziej, A. S. Sadullaev, B. I. Abullaev, L. H. Chinh,.... In 2005, Z. Blocki brought out the definition of function m - subharmonic $(SH_m(\Omega))$ and studied the solution of Hessian equation sole to this class, Then it followed that, in 2012, L. H. Chinh based on the ideas of Cegrell and brought out function classes $\mathcal{E}_m^0(\Omega)$, $\mathcal{F}_m(\Omega)$, $\mathcal{E}_m(\Omega)$ subclass of $SH_m(\Omega)$. These are unbounded m-subharmonic function classes but in which we can defined complex Hessian operator, the same to mentioned $\mathcal{E}^0(\Omega)$, $\mathcal{F}(\Omega)$, $\mathcal{E}(\Omega)$ of Cegrell. From that, the author proved its existence of complex m-Hessian operator $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ on $\mathcal{E}_m(\Omega)$ function. How do subextension and initial function control the m-Hessian measures? The study of these questions in this function remains a problem that need further studies.

The last problem mentioned in this thesis is the equation of complex Monge- Ampre for class Cegrell $\mathcal{N}(\Omega, f)$. The equation form is

$$(dd^c u)^n = F(u, .)d\mu,$$

. As we have known, the proof of the existence of weak solutions of this equation has been studied extensively by many authors. The majority of the results above has mentioned the case in which μ is a measure vanishing on pluripolar sets of Ω. In this paper, we would like to study weak solutions of Monge- Ampre for an arbitrary measure, in particular, for measures carried by a pluripolar set.

For these reasons, we have chosen the topic: **"Subextension of plurisubharmonic functions and applications"**.

2. The importance of the topic

As mentioned above, problems about subextension of plurisubharmonic functions in unbounded domains with boundary values have only appeared recently. Moreover, creating the connection between Monge - Ampre measures of subextension of plurisubharmonic function and given function has hardly been examined, except for the case of the class $\mathcal{F}(\Omega, f)$. Therefore, extending the problems in other classes is necessary and worth examining. The case is similar for the researching of m - subharmonic functions with the control of Hessian $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ and solving Monge-Ampre equations to find measures with values on pluripolar sets.

3. The aim of thesis

The aim of the thesis is to examine the subextension of plurisubharmonic functions in the class $\mathcal{E}_\chi(\Omega, f)$ where Ω is bounded hyperconvex in $\mathbb{C}^n$; class $\mathcal{F}(\Omega, f)$ with Ω - unbounded hyperconvex

in $\mathbb{C}^n$ and subextensions of m - subharmonic functions for the class $\mathcal{F}_m(\Omega)$ with Ω being bounded m- hyperconvex domains in $\mathbb{C}^n$. Moreover, the thesis is also proves the existenceof weak solutions of the equations of complex Monge - Ampère type in the class $\mathcal{N}(\Omega, f)$ for arbitrary measures, in particular, measures carried by pluripolar sets. We prove that problems about subextension in the classes $\mathcal{E}_\chi(\Omega, f)$, $\mathcal{F}_m(\Omega)$ with Ω being bounded hyperconvex domain and l m - hyperconvex domain come into effect. Besides, we also establish the equality between the Monge - Ampre measures of subextension functions and the given functions. Likewise, we create the existence of subextension in the class $\mathcal{F}(\Omega, f)$ when Ω is unbounded hyperconvex domain and the equality of measures is the same as mentioned above.

4. Study subjects

As we demonstrated in the reason for choosing the topic, the study subject of the thesis is the subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes, subextension of plurisubharmonic functions in unbounded hypercomplex and applications, subextension of m-subharmonic functions and equations of complex MongeAmpre type for arbitrary measures with conditions which are more general than previous studies of this problem. Furthermore, in cases we proposed to study, previous techniques and methods of other authors are not mentioned.

5. The meaning and practice of science thesis

The thesis helps to develop more deeply about the results of subextension of plurisubharmonic functions, subextension of m-subharmonic functions,weak solutions of equations of complex Monge - Ampère type for arbitrary measures. Methodically, the thesis helps diversify systems of tools and techniques of specialized studies, specifically applicated in the topic of the thesis and similar topics.

The thesis is one of the reference documents for Maters and Phd students doing the research.

6. Structure of the thesis

Structure of the thesis is demonstrated, following specific rules for thesis of Hanoi National University of Education, including beginning , overview- demonstrating history of the problem, analysis and judgemetn of the study of national and foreign authors realted to the thesis. The 4 remaining chapters of the thesis based on 4 other work, which has been uploaded and become public.

Chapter 1: Subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes.

Chapter 2: Subextension of plurisubharmonic functions in unbounded hyperconvex domains and applications.

Chapter 3: Subextension of m-subharmonic functions.

Chapter 4: Equations of complex Monge-Ampère type for arbitrary measures.

Finally, in the conclusion part, we review the results of his or her own thesis. This is the assertion that the stated idea of the topic of the thesis is true and the results reach the target. As a result, thesis has a number of contributions for specialized science, has scientific meaning and applications as mentioned in the beginning part, which is absolutely authentic. Simultaneously, in Recommendations part, we bravely propose a number of following study ideas to develop the topic of the thesis. We hope we could receive the attention and share from our colleagues to help us perfect the results of the research.

7. The location where the topic is discussed

Hanoi National University of Education.

Overview subextension of plurisubharmonic and Equations of complex Monge-Ampère type

1. Subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes

In the pluripotential theory, MongeAmpre operator is a tool served as thecenter and throughout the development of pluripotential theory. This operator is strongly researched since the second half of the XX century, in the way of describing subclasses of plurisubharmonic functions ($PSH(\Omega)$) that MongeAmpre operator is still defined as a continuous positive Radon measure on a decreasing sequence. In 1975, Y. Siu had shown that, $(dd^c u)^n$ cannot be defined as a regular Borel measure as plurisubharmonic functions with any u. In 1982, Bedford and Taylor had defined $(dd^c)^n$ operator on a class of local bounded plurisubharmonic functions, $PSH(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. Other fundalmental results about the pluripotential theory related to this problem can be found in documentaries. To continue the way of extending defined domain of MongeAmpre complex operator mentioned, in 1998, 2004 and 2008, in his work, Cegrell had described many subclasses of $\mathcal{PSH}(\Omega)$ with Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, in which $\mathcal{E}(\Omega)$ is the biggest class that MongeAmpre operator can still be defined as a Radon measure, simutanuously this operator is continuous on a decreasing sequence of a plurisulharmonic function. This means that if $u \in \mathcal{E}(\Omega)$ then $(dd^c u)^n$ exists and if $\{u_j\} \subset \mathcal{E}(\Omega)$ with $u_j \searrow u$ then $(dd^c u_j)^n$ weakly converges to $(dd^c u)^n$. In the beginning of the thesis we study the problem of the subextension of plurisubharmonic functions with boundary values in pluricomplex energy classes weighted $\mathcal{E}_\chi(\Omega, f)$.

The problem of subextension of plurisubharmonic functions has been concerned since the 80 of the previous century. El Mir gave in 1980 an example of a plurisubharmonic function on the unit bidisc for which the restriction to any smaller bidisc admits no subextension to the whole space. In 1987, Fornaess and Sibony poited out that for a ring domain in $\mathbb{C}^2$, there exists a plurisubharmonic function which admits no subextension inside the hole. In 1988, Bedford and Taylor proved that any smoothly bounded domain in $\mathbb{C}^n$ is a domain of existence of a smooth plurisubharmonic function.

We define the following subclasses of $PSH^-(\Omega)$ on set Ω is a bounded hyperconvex domain in $\mathbb{C}^n$:

Definition 1

$$\mathcal{E}_0(\Omega) = \big\{\varphi \in PSH^-(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} \varphi(z) = 0, \int_\Omega (dd^c\varphi)^n < \infty\big\},$$

$$\mathcal{E}(\Omega) = \big\{\varphi \in PSH^-(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighbourhhood } U \ni z_0,$$
$$\mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi \text{ on } U,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\big\},$$

$$\mathcal{F}(\Omega) = \big\{\varphi \in PSH^-(\Omega) : \exists\ \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\big\},$$

$$\mathcal{F}^a(\Omega) = \big\{\varphi \in \mathcal{F}(\Omega) : (dd^c\varphi)^n(E) = 0, \forall E \subset \Omega \text{ pluripola set}\big\},$$

now for each p > 0, put

$$\mathcal{E}_p(\Omega) = \big\{\varphi \in PSH^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi, \sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < \infty\big\}.$$

Remark: The following inclusions are obvious $\mathcal{E}_0(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$.

On bounded hyperconvex domains in $\mathbb{C}^n$, Cegrell and Zeriahi investigated the subextension problem for the class $\mathcal{F}(\Omega)$. In 2013, the authors proved that if $\Omega \Subset \widetilde{\Omega}$ are bounded hyperconvex domains in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$, then there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω and

$$\int_{\widetilde{\Omega}} (dd^c\widetilde{u})^n \leq \int_\Omega (dd^c u)^n.$$

In the class $\mathcal{E}_p(\Omega), p > 0$, the subextension problem was investigated by P. H. Hiep. He proved that if $\Omega \subset \widetilde{\Omega} \Subset \mathbb{C}^n$ are bounded hyperconvex domains and $u \in \mathcal{E}_p(\Omega)$, then there exists a function $\widetilde{u} \in \mathcal{E}_p(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω and

$$\int_{\widetilde{\Omega}} (-\widetilde{u})^p (dd^c\widetilde{u})^n \leq \int_\Omega (-u)^p (dd^c u)^n.$$

In here, The author had proved the condition of Ω compact relatively in $\widetilde{\Omega}$ to be superfluous.

Recently a weighted pluricomplex energy class $\mathcal{E}_\chi(\Omega)$, which is generalization of the classes $\mathcal{E}_p(\Omega)$ and $\mathcal{F}(\Omega)$ was introduced and investigated by Benelkourchi, Guedj and Zeriahi. Benelkourchi studied subextension for the class $\mathcal{E}_\chi(\Omega)$. Benelkourchi claimed that if $\Omega \subset \widetilde{\Omega}$ are hyperconvex domains in $\mathbb{C}^n$ and $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ is a decreasing function with $\chi(-\infty) = +\infty$ then for every $u \in \mathcal{E}_\chi(\Omega)$ there exists $\widetilde{u} \in \mathcal{E}_\chi(\widetilde{\Omega})$ such that $\widetilde{u} \leq u$ on Ω and $(dd^c\widetilde{u})^n \leq (dd^c u)^n$ on Ω and

$$\int_{\widetilde{\Omega}} \chi(\widetilde{u})(dd^c\widetilde{u})^n \leq \int_\Omega \chi(u)(dd^c u)^n.$$

If we take $\chi(t) = (-t)^p, p > 0$ then the class $\mathcal{E}_\chi(\Omega)$ coincides with the class $\mathcal{E}_p(\Omega)$. If $\chi(t)$ is bounded and $\chi(0) > 0$ then $\mathcal{E}_\chi(\Omega)$ is the class $\mathcal{F}(\Omega)$ and then the results of subextension turn back to the results mentioned above.

The subextension problem in the classes with boundary values was considered in recent years. Namely, in 2008, Czyż and Hed showed that if Ω and $\widetilde{\Omega}$ are two bounded hyperconvex domains such that $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}^n, n \geq 1$ and $u \in \mathcal{F}(\Omega, f)$ with $f \in \mathcal{E}(\Omega)$ has subextension $v \in \mathcal{F}(\widetilde{\Omega}, g)$ with $g \in \mathcal{E}(\widetilde{\Omega}) \cap MPSH(\widetilde{\Omega})$, and

$$\int_{\widetilde{\Omega}} (dd^c v)^n \leq \int_\Omega (dd^c u)^n,$$

under the assumption that $f \geq g$ on Ω, where $MPSH(\widetilde{\Omega})$ denotes the set of maximal plurisubharmonic functions on $\widetilde{\Omega}$.
It should be noticed that in results above only estimation of total Monge-Ampère mass of subextension was obtained. In 2014, L.M.Hai anh N.X.Hong investigated subextension in the class $\mathcal{F}(\Omega, f)$ and They proved that the Monge-Ampère measure of subextension does not change. Namely let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$ and $g \in \mathcal{E}(\widetilde{\Omega}) \cap MPSH(\widetilde{\Omega})$ with $f \geq g$ on Ω, then for every $u \in \mathcal{F}(\Omega, f)$ with

$$\int\limits_{\Omega} (dd^c u)^n < +\infty,$$

there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $\widetilde{u} \leq u$ on Ω and $(dd^c\widetilde{u})^n = 1_\Omega (dd^c u)^n$on $\widetilde{\Omega}$.

In this chapter we extend this result to the class $\mathcal{E}_\chi(\Omega, f)$. Our main theorem is the following.
Theorem 1.2.1. *Let $\Omega \Subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$, $g \in \mathcal{E}(\widetilde{\Omega}) \cap MPSH(\widetilde{\Omega})$ with $f \geq g$ on Ω. Assume that $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ is a decreasing continuous function such that $\chi(t) > 0$ for all $t < 0$. Then for every $u \in \mathcal{E}_\chi(\Omega, f)$ such that*

$$\int\limits_{\Omega} [\chi(u) - \rho](dd^c u)^n < +\infty,$$

for some $\rho \in \mathcal{E}_0(\Omega)$, there exists $\widetilde{u} \in \mathcal{E}_\chi(\widetilde{\Omega}, g)$ such that $\widetilde{u} \leq u$ on Ω and

$$\chi(\widetilde{u})(dd^c\widetilde{u})^n = 1_\Omega \chi(u)(dd^c u)^n \text{ on } \widetilde{\Omega}.$$

2. Subextension of plurisubharmonic functions in unbounded hyperconvex domains and applications

In the paper we study subextension of plurisubharmonic functions for the class $\mathcal{F}(\Omega, f)$ introduced and investigated in paper "The complex Monge-Ampère equation in unbounded hyperconvex domains in $\mathbb{C}^n$" on unbounded hyperconvex domains Ω in $\mathbb{C}^n$. For the history and results on subextension of plurisubharmonic functions in the Cegrell classes on bounded hyperconvex domains in $\mathbb{C}^n$ we refer readers to our earlier **1**. Note that to study subextension of plurisubharmonic functions on domains in $\mathbb{C}^n$ closely concerns with the solvability of the complex Monge-Ampère equations on them. Hence, up to now, subextension of plurisubharmonic functions only is carried out on bounded hyperconvex domains in $\mathbb{C}^n$ because for these domains ones obtains many perfect results on the solvability of the complex Monge-Ampère equations. However, it is quite difficult when we want to consider this problem for unbounded hyperconvex domains in $\mathbb{C}^n$ because results on the solvability of the complex Monge-Ampère on them are limited.

Relying on our some recent results for solving the complex Monge-Ampère equations on unbounded hyperconvex domains in $\mathbb{C}^n$ introduced and investigated by L.M.Hai- N.V.Trao and N.X.Hong in *The complex MongeAmpre equation in unbounded hyperconvex domains in* $\mathbb{C}^n$. We recall the definition of the Cegrell classes for unbounded hyperconvex domains which were introduced in L.M.Hai-N.V.Trao and N.X.Hong.
Definition 2 Let Ω be a unbounded hyperconvex domain in $\mathbb{C}^n$ such that $PSH^s(\Omega) \cap L^\infty(\Omega) \neq \varnothing$. Put

$$\mathcal{E}_0(\Omega) = \{u \in PSH^-(\Omega) \cap L^\infty(\Omega) : \forall \ \ \varepsilon > 0, \ \ \{u < -\varepsilon\} \Subset \Omega, \ \int\limits_{\Omega} (dd^c u)^n < \infty\},$$

$$\mathcal{F}(\Omega) = \big\{u \in PSH^-(\Omega) : \exists \ \ \mathcal{E}_0(\Omega) \ni u_j \searrow u, \ \ \sup_j \int_\Omega (dd^c u_j)^n < \infty\big\},$$

and

$$\mathcal{E}(\Omega) = \big\{u \in PSH^-(\Omega) : \forall \ \ U \Subset \Omega, \exists \ \ v \in \mathcal{F}(\Omega) \ \ \text{vi} \ \ v = u \ \ \text{trong} \ \ U\big\}.$$

If $f \in MPSH^-(\Omega) \cap \mathcal{C}(\Omega)$ and $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}, \mathcal{E}\}$ then we put

$$\mathcal{K}(\Omega, f) = \{u \in PSH^-(\Omega) : \exists \ \ \psi \in \mathcal{K}(\Omega), \ \psi + f \leq u \leq f \ \text{trong} \ \Omega\}.$$

Remark It is clear that $\mathcal{E}_0(\Omega, f) \subset \mathcal{F}(\Omega, f) \subset \mathcal{E}(\Omega, f)$.

We will extend our result in L.M.Hai and N.X.Hong for unbounded hyperconvex domains in $\mathbb{C}^n$. The first main result is the following theorem.

Theorem 2.2.1. *Let $\Omega \subset \widetilde{\Omega}$ be unbounded hyperconvex domains in $\mathbb{C}^n$ such that $PSH^s(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega}) \neq \varnothing$. Then for every $f \in MPSH^-(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ and for every $u \in \mathcal{F}(\Omega, f)$ such that*

$$\int_\Omega (dd^c u)^n < \infty,$$

there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega}, f)$ such that $\widetilde{u} \leq u$ on Ω and

$$(dd^c \widetilde{u})^n = 1_\Omega (dd^c u)^n \ \ \text{on} \ \widetilde{\Omega}.$$

As an application of the above result, in the next section of the chapter we study approximation of plurisubharmonic functions by an increasing sequence of plurisubharmonic functions defined on larger domains. Let $\Omega \Subset \Omega_{j+1} \Subset \Omega_j$ be bounded hyperconvex domains in $\mathbb{C}^n$. In 2006, Benelkourchi proved that if $\lim_{j\to\infty} Cap(K, \Omega_j) = Cap(K, \Omega)$, for all compact subset $K \Subset \Omega$ then for every $u \in \mathcal{F}^a(\Omega)$ there exists an increasing sequence of functions $u_j \in \mathcal{F}^a(\Omega_j)$ such that $u_j \longrightarrow u$ a.e. in Ω. Next, in 2008, in order to improve the above result of Benelkourchi, Cegrell and Hed proved that if there exists $v \in \mathcal{N}(\Omega)$, $v < 0$ and $v_j \in \mathcal{N}(\Omega_j)$ such that $v_j \longrightarrow v$ a.e. on Ω then for every $u \in \mathcal{F}(\Omega)$ there exists an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j)$ such that $u_j \longrightarrow u$ a.e. in Ω.
In 2010, Hed investigated the above result for the class $\mathcal{F}(\Omega, f)$. Namely she proved that if there exists $v \in \mathcal{N}(\Omega)$, $v < 0$ and $v_j \in \mathcal{N}(\Omega_j)$ such that $v_j \longrightarrow v$ a.e. on Ω then for every $f \in MPSH^-(\Omega_1) \cap \mathcal{C}(\Omega_1)$ and $u \in \mathcal{F}(\Omega, f)$ such that

$$\int_\Omega (dd^c u)^n < \infty,$$

there exists an increasing sequence of functions $u_j \in \mathcal{F}(\Omega_j, f)$ such that $u_j \longrightarrow u$ a.e. in Ω. In this chapter, by using an another approach, we prove the above result of Hed for unbounded hyperconvex domains in $\mathbb{C}^n$. Namely, we prove the following theorem.

Theorem 2.3.1. *Let Ω be a unbounded hyperconvex domain in $\mathbb{C}^n$ and let $\{\Omega_j\}_{j=1}^\infty$ be a sequence of unbounded hyperconvex domains such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $PSH^s(\Omega_1) \cap L^\infty(\Omega_1) \neq \varnothing$. Assume that there exist $\psi \in \mathcal{F}(\Omega)$ and $\psi_j \in \mathcal{F}(\Omega_j)$ such that $\psi_j \nearrow \psi$ a.e in Ω as $j \nearrow \infty$. Then for every $f \in MPSH^-(\Omega_1) \cap \mathcal{C}(\Omega_1)$ and for every $u \in \mathcal{F}(\Omega, f)$ such that*

$$\int_\Omega (dd^c u)^n < \infty,$$

there exists $u_j \in \mathcal{F}(\Omega_j, f)$ such that $u_j \nearrow u$ a.e. in Ω as $j \nearrow \infty$.

3. Subextension of m-subharmonic functions

In recent times, the extention the class of plurisubharmonic functions and to study a class of the complex differential operators more general than the Monge-Ampère operator have been studying by many authors, such as Z. Błocki, S. Dinew, Kołodziej, A. S. Sadullaev, B. I. Abullaev, L. H. Chinh, ... They introduced m-subharmonic functions and studied the complex Hessian operator. The results of Z. Błocki, S. Dinew, Kołodziej, A. S. Sadullaev were mainly about on locally bounded $m-$subharmonic functions. Continuing to study the complex Hessian operator for m-subharmonic functions which may be not locally bounded, in recent preprint, L. H. Chinh introduced the Cegrell classes $\mathcal{E}_m^0(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ associated to m-subharmonic functions. However, it is difficult to image this class. Thus, the problem for us is studying the class $\mathcal{E}_m(\Omega)$ more detail or describe and giving some characterrizations of this class.

In the following section we study the problem of subextension for the class m- unbounded plurisubharmonic function, in particular for class $\mathcal{F}_m(\Omega)$.
Subextension for the class $\mathcal{F}_m(\Omega)$ in the case Ω is a hyperconvex domain in $\mathbb{C}^n$ was studied earlier. However, the result on subextension which the author obtained in the class $\mathcal{F}_m(\Omega)$ in the above mentioned paper is limited. Firstly, the author has to assume that Ω is a relatively compact hyperconvex domain in $\widetilde{\Omega}$. Secondly, the author does not give a control of the complex Hessian measures of subextension and given m-subharmonic function. In this note we try to overcome the above limits. We prove the existence of subextension for the class $\mathcal{F}_m(\Omega)$ in the case $\Omega, \widetilde{\Omega}$ are bounded m-hyperconvex domains in $\mathbb{C}^n$ without assymption that Ω is relatively compact in $\widetilde{\Omega}$ and to control the complex Hessian measure of subextension. Namely we prove the following.
Theorem 3.2.1. *Let $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}^n$ be bounded m-hyperconvex domains and $u \in \mathcal{F}_m(\Omega)$. Then there exists $w \in \mathcal{F}_m(\widetilde{\Omega})$ such that $w \leq u$ on Ω and*

$$(dd^c w)^m \wedge \beta^{n-m} = 1_\Omega (dd^c u)^m \wedge \beta^{n-m}.$$

From the above theorem, we obtain the following corollary.
Corollary 3.2.5. *Let $\Omega \subset \widetilde{\Omega}$ be bounded m-hyperconvex domains and $\{u_j\}_{j\geq 1}, u \subset \mathcal{F}_m(\Omega)$ be such that $u_j \geq u$, u_j is convergent in C_m-capacity to u on Ω. Assume that $\widetilde{u}_j, \widetilde{u}$ are subextensions of u_j, u, respectively, to $\widetilde{\Omega}$. Then $H_m(\widetilde{u}_j)$ is weakly convergent to $H_m(\widetilde{u})$ on $\widetilde{\Omega}$.*

4. Equations of complex Monge-Ampère type for arbitrary measures

In the pluripotential theory, finding solutions to Dirichler problem

$$\begin{cases} u \in PSH(\Omega) \cap L^\infty(\Omega) \\ (dd^c u)^n = d\mu \\ \lim\limits_{z \to x} u(z) = \varphi(x), \forall x \in \partial\Omega. \end{cases} \tag{1}$$

in which Ω is an open set, bounded in $\mathbb{C}^n$, μ is a positive Borel measure on Ω and $\varphi \in C(\partial\Omega)$ is a continuous function, always draws attentions of many authors. In case $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain and $d\mu = f dV_{2n}$, $f \in C(\overline{\Omega})$ then Bedford - Taylor proved (1) to have an unique solution. If $d\mu = f dV_{2n}$, $f \in C^\infty(\overline{\Omega})$, $f > 0$ and $\partial\Omega$ is smooth, the authors proved (1) to have an unique solution $u \in C^\infty(\Omega)$. One way to solve the problem is to examine the existence of the solution of the equation above if we can prove the existence at a subsolution. In 1995, S. Kolodziej proved that in a strictly pseudoconvex $\Omega \subset \mathbb{C}^n$: if there exists a subsolution in the class of bounded plurisubharmonic function then equation (1) has a bounded solution. In 2009, Åhag, Cegrell, Czyż and H. Hiep researched the problem in a hyperconvex domain with the class of unnecessarily bounded plurisubharmonic with the extending boundary values and resulted in:

Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain and $H \in \mathcal{E}(\Omega) \cap MSHP(\Omega)$. If there are $w \in \mathcal{E}(\Omega)$ so that $\mu \leq (dd^c u)^n$ then $\exists u \in \mathcal{E}(\Omega, H)$ with $(dd^c u)^n = \mu$. To continue the research of solving Monge - Ampère equation, in chapter 4 of the thesis we discuss the weak solutions of equation of complex Monge - Ampère type. The equation of the form

$$(dd^c u)^n = F(u, .)d\mu, \tag{2}$$

The proof of the existence of weak solutions of this equation has been investigated by many authors; When μ vanishes on all pluripolar sets and $\mu(\Omega) < +\infty$, F is bounded by an integrable function for μ which is independent of the first variable then for all $f \in MPSH(\Omega) \cap \mathcal{E}(\Omega)$, Cegrell and Kołodziej proved that (2) has a solution $u \in \mathcal{F}^a(\Omega, f)$ where $MPSH(\Omega)$ denotes the set of maximal plurisubharmonic functions and $\mathcal{F}^a(\Omega, f)$ is the set of plurisubharmonic functions introduced and investigated by Cegrell. Next, Czyż investigated the equation (2) in the class $\mathcal{N}(\Omega, f)$. He proved that if μ vanishes on pluripolar sets of Ω, F is a continuous function of the first variable and bounded by an integrable function for $(-\varphi)\mu$ which is independent of the first variable then the equation (2) is solvable in the class $\mathcal{N}(\Omega, f)$. More recently, under the same assumption that μ vanishes on all pluripolar sets of Ω and there exists a subsolution $v_0 \in \mathcal{N}^a(\Omega)$, i.e there exists a function $v_0 \in \mathcal{N}^a(\Omega)$ such that $(dd^c v_0)^n \geq F(v_0, .)d\mu$, Benelkourchi showed that (2) has a solution $u \in \mathcal{N}^a(\Omega, f)$.

The problem here is that we want to study weak solutions of the equation (2) for an arbitrary measure, in particular, for measures carried by a pluripolar set. The main result is the following.

Theorem 4.2.1. *Let Ω be a bounded hyperconvex domain and μ be a nonnegative measure in Ω. Assume that $F : \mathbb{R} \times \Omega \longrightarrow (0, +\infty)$ is a $dt \times d\mu$-measurable function such that:*

(1) For all $z \in \Omega$, the function $t \longmapsto F(t, z)$ is continuous and nondecreasing.

(2) For all $t \in \mathbb{R}$, the function $z \longmapsto F(t, z)$ belongs to $L^1_{loc}(\Omega, \mu)$.

(3) There exists a function $w \in \mathcal{N}(\Omega)$ such that $(dd^c w)^n \geq F(w, .)d\mu$.

Then for any maximal plurisubharmonic function $f \in \mathcal{E}(\Omega)$ there exists $u \in \mathcal{N}(\Omega, f)$ such that $u \geq w$ and $(dd^c u)^n = F(u, .)d\mu$ in Ω.

Chapter 1

Subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes

As in the introduction. The purpose of this project is to prove the subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes in $\mathcal{E}_\chi(\Omega, f)$.

Chapter 1 includes two parts. The first part, important background knowledge for this chapter and the following ones is presented. While the demonstration of the main theorem is displayed in the second chapter.

The results were pulled out from the article[1] (in the mentioned project category of the thesis)

1.1 Some definitions and consequences

Let Ω be opening set in $\mathbb{C}^n$. By $PSH(\Omega)$ we denote in turn the set of plurisubharmonic functions on Ω and By $PSH^-(\Omega)$ and the set of negative maximal plurisubharmonic functions on Ω.

Definition 1.1.1. *Let $\Omega \subset \widetilde{\Omega}$ be domains in $\mathbb{C}^n$ and let u be a plurisubharmonic function on Ω (briefly, $u \in PSH(\Omega)$). A function $\widetilde{u} \in PSH(\widetilde{\Omega})$ is subextension of u if for all $z \in \Omega, \widetilde{u}(z) \leq u(z)$.*

Remark 1.1.2. If $\widetilde{u}$ is subextension of u, at the point $z \in \Omega$ so that $u(z) = -\infty$ then $\widetilde{u}(z) = -\infty$.

Definition 1.1.3. *Set open Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ if Ω is a bounded domain in $\mathbb{C}^n$ and there exists a plurisubharmonic function $\varphi : \Omega \longrightarrow (-\infty, 0)$ such that for every $c < 0$ the set $\Omega_c = \{z \in \Omega : \varphi(z) < c\} \Subset \Omega$.*

Definition 1.1.4. *A plurisubharmonic function u on Ω is said to be maximal (briefly, $u \in MPSH(\Omega)$) if for every compact set $K \Subset \Omega$ and for every $v \in PSH(\Omega)$, if $v \leq u$ on $\Omega \setminus K$ then $v \leq u$ on Ω.*

By $MPSH^-(\Omega)$ we denote the set of negative maximal plurisubharmonic functions on Ω

Remark 1.1.5. It is well known that locally bounded plurisubharmonic functions are maximal if and only if they satisfy the homogeneous Monge-Ampère equation $(dd^c u)^n = 0$. Błocki extended the above result for the class $\mathcal{E}(\Omega)$.

We recall the class $\mathcal{N}(\Omega)$ introduced by Cegren.

Definition 1.1.6. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $\{\Omega_j\}_{j\geq 1}$ a fundamental sequence of Ω. This is an increasing sequence of strictly pseudoconvex subsets $\{\Omega_j\}_{j\geq 1}$ of Ω such that $\Omega_j \Subset \Omega_{j+1}$ and $\bigcup_{j=1}^{+\infty} \Omega_j = \Omega$.*

Let $\varphi \in PSH^-(\Omega)$. For each $j \geq 1$, put

$$\varphi^j = \sup\{u : u \in PSH(\Omega), \quad u \leq \varphi \ \text{ on } \ \Omega\backslash\overline{\Omega}_j\}.$$

The function $(\lim_{j\to\infty} \varphi^j)^* \in MPSH(\Omega)$. Set

$$\mathcal{N}(\Omega) = \{\varphi \in \mathcal{E}(\Omega) : \varphi^j \uparrow 0\}.$$

Remark 1.1.7. It is easy to see that $\mathcal{F}(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}(\Omega)$.

Next, we recall the class $\mathcal{E}_\chi(\Omega)$ and the relation between this class and the classes $\mathcal{E}_p(\Omega), \mathcal{F}(\Omega)$ and $\mathcal{N}(\Omega)$.

Definition 1.1.8. *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function and Ω be a hyperconvex domain in $\mathbb{C}^n$. We say that the function $u \in PSH^-(\Omega)$ belongs to $\mathcal{E}_\chi(\Omega)$ if there exists a sequence $\{u_j\} \subset \mathcal{E}_0(\Omega)$ decreasing to u on Ω and satisfying*

$$\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^n < +\infty.$$

Remark 1.1.9.
a) If we take $\chi(t) = (-t)^p, p > 0$ then the class $\mathcal{E}_\chi(\Omega)$ coincides with the class $\mathcal{E}_p(\Omega)$.
b) If $\chi(t)$ is bounded and $\chi(0) > 0$ then $\mathcal{E}_\chi(\Omega)$ is the class $\mathcal{F}(\Omega)$.
c) Corollary 3.3 in L.M.Hai anh P.H.Hiep claims that if $\chi \not\equiv 0$ then $\mathcal{E}_\chi(\Omega) \subset \mathcal{E}(\Omega)$ and, hence, in this case the Monge-Ampère operator is well defined on $\mathcal{E}_\chi(\Omega)$.
d) Corollary 3.3 in L.M.Hai anh P.H.Hiep shows that if $\chi(t) > 0$ for $t < 0$ then $\mathcal{E}_\chi(\Omega) \subset \mathcal{N}(\Omega)$. Moreover, Theorem 2.7 in Benelkouchri(2011) implies that

$$\mathcal{E}_\chi(\Omega) = \left\{ u \in \mathcal{N}(\Omega) : \int_\Omega \chi(u)(dd^c u)^n < +\infty \right\}.$$

In this thesis, we are supposed to use the concept as follows.

Definition 1.1.10. *Let $\Omega \subset \mathbb{C}^n$ be an opening set, μ a positive Borel measure on Ω, assume that:*
i) μ vanishes on pluripolar sets of Ω for all $A \subset \Omega$, A is a pluripolar set, we have $\mu(A) = 0$.
ii) μ is carried by a pluripolar set if $A \subset \Omega$ exists(A is a pluripolar set), so that $\mu(A) = \mu(\Omega)$. In this case we can write down $\mu = 1_A\mu$.

Next, We recall classes of plurisubharmonic functions with generalized boundary values in the class $\mathcal{E}(\Omega)$.

Definition 1.1.11. *Let $\mathcal{K} \in \{\mathcal{E}_0(\Omega), \mathcal{F}(\Omega), \mathcal{N}(\Omega), \mathcal{E}_\chi(\Omega), \mathcal{E}(\Omega)\}$ and let $f \in \mathcal{E}(\Omega)$. Then we say that a plurisubharmonic function u defined on Ω is in $\mathcal{K}(\Omega, f)$ if there exists a function $\varphi \in \mathcal{K}$ such that*

$$\varphi + f \leq u \leq f,$$

on Ω. By $\mathcal{K}^a(\Omega, f)$ we denote the set of plurisubharmonic functions $u \in \mathcal{K}(\Omega, f)$ such that $(dd^c u)^n$ vanishes on all pluripolar sets of Ω.

We need the following proposition which will be used in the main result.

Proposition 1.1.12. *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing continuous function such that $\chi(t) > 0$ for all $t < 0$ and Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Assume that μ is a positive Radon measure which vanishes on pluripolar sets of Ω and $u, v \in \mathcal{E}(\Omega)$ are such that $\chi(u)(dd^c u)^n \geq \mu$ and $\chi(v)(dd^c v)^n \geq \mu$. Then*

$$\chi(\max(u, v))(dd^c \max(u, v))^n \geq \mu.$$

Proposition 1.1.13. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$. Then for every $u \in \mathcal{N}(\Omega, f)$ such that*

$$\int\limits_{\{u=-\infty\}\cap\Omega} (dd^c u)^n < +\infty,$$

there exists $v \in \mathcal{F}(\Omega, f)$ such that $v \geq u$ and

$$(dd^c v)^n = 1_{\{u=-\infty\}}(dd^c u)^n.$$

1.2 Subextension of plurisubharmonic functions in classes $\mathcal{E}_\chi(\Omega, f)$

In this section we give the main result of the chapter. However, to arrive at the desired result we need some auxiliary lemmas

Theorem 1.2.1. *Let $\Omega \Subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$, $g \in \mathcal{E}(\widetilde{\Omega}) \cap MPSH(\widetilde{\Omega})$ with $f \geq g$ on Ω. Assume that $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ is a decreasing continuous function such that $\chi(t) > 0$ for all $t < 0$. Then for every $u \in \mathcal{E}_\chi(\Omega, f)$ such that*

$$\int\limits_{\Omega} [\chi(u) - \rho](dd^c u)^n < +\infty,$$

for some $\rho \in \mathcal{E}_0(\Omega)$, there exists $\widetilde{u} \in \mathcal{E}_\chi(\widetilde{\Omega}, g)$ such that $\widetilde{u} \leq u$ on Ω and

$$\chi(\widetilde{u})(dd^c \widetilde{u})^n = 1_\Omega \chi(u)(dd^c u)^n \text{ on } \widetilde{\Omega}.$$

we need some auxiliary lemmas.

Lemma 1.2.2. *Let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $g \in \mathcal{E}(\widetilde{\Omega}) \cap MPSH(\widetilde{\Omega})$ with $f \geq g$ on Ω. Assume that $u \in \mathcal{F}(\Omega, f)$ is such that*

(a) $(dd^c u)^n$ is carried by a pluripolar set.

(b) $\int\limits_{\Omega} (dd^c u)^n < +\infty$.

Then the function

$$\widetilde{u} := (\sup\{\varphi \in \mathcal{F}(\widetilde{\Omega}, g) : \varphi \leq u \text{ on } \Omega\})^*$$

belongs to $\mathcal{F}(\widetilde{\Omega}, g)$ and $(dd^c \widetilde{u})^n = 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$.

Lemma 1.2.3. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let μ be a positive Radon measure which vanishes on pluripolar sets of Ω with $\mu(\Omega) < +\infty$. Let $\chi : \mathbb{R}^- \to \mathbb{R}^+$ be a bounded decreasing continuous function such that $\chi(t) > 0$ for all $t < 0$ and $\chi(-\infty) = 1$. Assume that $f \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$ and $v \in \mathcal{F}(\Omega, f)$ such that $(dd^c v)^n$ is carried by a pluripolar set and*

$$\int\limits_{\Omega} (dd^c v)^n < +\infty.$$

Then the function u defined by

$$u := (\sup\{\varphi \in \mathcal{E}(\Omega) : \varphi \leq v \text{ and } \chi(\varphi)(dd^c\varphi)^n \geq \mu\})^*$$

belongs to $\mathcal{N}(\Omega, f)$ and

$$\chi(u)(dd^c u)^n \geq \mu + (dd^c v)^n.$$

Moreover, if $supp(dd^c v)^n \Subset \Omega$ and $\int\limits_\Omega (-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$ then

$$\chi(u)(dd^c u)^n = \mu + (dd^c v)^n.$$

Chapter 2

Subextension of plurisubharmonic functions in unbounded hyperconvex domains and applications

As in the introduciton, we will present the subextension of plurisubharmonic functions in the classes $\mathcal{F}(\Omega, f)$ with Ω being unbounded hyperconvex domains. In the application par, we solve approximate problems bout plurisubharmonic functions with boundary values in unbounded hyperconvex $\mathbb{C}^n$.

Chapter 2 includes three parts. In the first chapter,some definitions and important clauses are presented for later demonstration. Some lemmas and the main theorem are displayed in the second one. The application is in the third part. Here, we apply the result of subextension of functions in unbounded hyperconvex domains to approximate problems about plurisubharmonic functions in increasing sequence of plurisubharmonic functions in wider domains.

Chapter 2 was based on the article [2] (in the mentioned project category of the thesis).

2.1 Some definitions and consequences

Definition 2.1.1. *Let Ω be a domain in $\mathbb{C}^n$. A negative plurisubharmonic function $u \in PSH^-(\Omega)$ is called to be strictly plurisubharmonic if for all $U \Subset \Omega$ there exists $\lambda > 0$ such that the function $u(z) - \lambda|z|^2 \in PSH(U)$. That is $dd^c u \geq 4\lambda\beta$ on U, where $\beta = \frac{i}{2}\sum\limits_{j=1}^{n} dz_j \wedge d\bar{z}_j$ is the canonical Kähler form in $\mathbb{C}^n$.*

By $PSH^s(\Omega)$ denotes the set of all negative strictly plurisubharmonic functions in Ω.
In the article "The complex MongeAmpre equation in unbounded hyperconvex domains in $\mathbb{C}^n$" of L.M.Hai, N.V.Trao, N.X.Hong(2014) example 3.2 has shown to prove the existence of an unbounded hyperconvex domain Ω in $\mathbb{C}^n$ such that $PSH^s(\Omega) \cap L^\infty(\Omega) \neq \varnothing$.

Example 2.1.2. Let $n \geq 1$ be an interger. Put $\rho(z) := 12|\Re z_1|^2 - \frac{1}{(|z_1|^2+1)^2} + \sum_{j=2}^{n} |z_j|^2$, where $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n, z_j = x_j + iy_j, j = 1, \ldots, n$. Let Ω be a connected component of the open set

$$\{z \in \mathbb{C}^n : \rho(z) < 0\},$$

that contains the line $(iy_1, 0)$, $y_1 \in \mathbb{R}$. It is easy to see that Ω is an unbounded domain in $\mathbb{C}^n$ and ρ is a strictly plurisubharmonic function on Ω (see Example 3.2 in L.M.Hai, N.X.Hong, N.V.Trao).

Moreover, throughout the chapter we always keep the assumption that $PSH^s(\Omega)\cap L^\infty(\Omega)\neq\varnothing$ because under this assumption, in the case Ω is a unbounded hyperconvex domain in $\mathbb{C}^n$, Proposition 4.2 in in L.M.Hai-N.V.Trao and N.X.Hong implies that if $u\in\mathcal{E}(\Omega,f)$ then $u\in\mathcal{E}(D)$ for every bounded hyperconvex domain $D\Subset\Omega$ and, hence, in this case the complex Monge-Ampère operator $(dd^c.)^n$ is well defined in the class $\mathcal{E}(\Omega,f)$.
Now, we give some results concerning to the class $\mathcal{F}(\Omega,f)$ when Ω is a unbounded hyperconvex domain in $\mathbb{C}^n$.

Proposition 2.1.3. *Let Ω be a unbounded hyperconvex domain in $\mathbb{C}^n$ such that $PSH^s(\Omega)\cap L^\infty(\Omega)\neq\varnothing$ and let $f\in MPSH^-(\Omega)\cap\mathcal{C}(\Omega)$. Assume that $u,v\in\mathcal{F}(\Omega,f)$. Then the following hold.*

(a) If $u\le v$ then

$$\int_\Omega (dd^cu)^n\ge\int_\Omega (dd^cv)^n.$$

(b) If $u\le v$, $(dd^cu)^n\le(dd^cv)^n$ and $\int_\Omega(dd^cu)^n<\infty$ then $u=v$.

2.2 Subextension of plurisubharmonic functions in unbounded hyperconvex domains

The first main result is the following theorem.

Theorem 2.2.1. *Let $\Omega\subset\widetilde{\Omega}$ be unbounded hyperconvex domains in $\mathbb{C}^n$ such that $PSH^s(\widetilde{\Omega})\cap L^\infty(\widetilde{\Omega})\neq\varnothing$. Then for every $f\in MPSH^-(\widetilde{\Omega})\cap\mathcal{C}(\widetilde{\Omega})$ and for every $u\in\mathcal{F}(\Omega,f)$ such that*

$$\int_\Omega (dd^cu)^n<\infty,$$

there exists $\widetilde{u}\in\mathcal{F}(\widetilde{\Omega},f)$ such that $\widetilde{u}\le u$ on Ω and

$$(dd^c\widetilde{u})^n=1_\Omega(dd^cu)^n \text{ on } \widetilde{\Omega}.$$

we first need some following auxiliary results:

Lemma 2.2.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f\in MPSH^-(\Omega)\cap\mathcal{E}(\Omega)$. Assume that $w\in\mathcal{E}(\Omega)$ and μ is a positive Borel measure in Ω such that $w\le f$ in Ω, $\mu\le(dd^cw)^n$ in Ω and $\int_\Omega(dd^cw)^n<\infty$. Then there exists $u\in\mathcal{F}(\Omega,f)$ such that $u\ge w$ and $(dd^cu)^n=\mu$ in Ω.*

Lemma 2.2.3. *Let Ω be a unbounded hyperconvex domain in $\mathbb{C}^n$ such that $PSH^s(\Omega)\cap L^\infty(\Omega)\neq\varnothing$ and let $f\in MPSH^-(\Omega)\cap\mathcal{C}(\Omega)$. Assume that $\{\Omega_j\}_{j=1}^\infty$ is a sequence of bounded hyperconvex domains such that $\Omega_j\Subset\Omega_{j+1}\Subset\Omega$ and $\Omega=\bigcup\limits_{j=1}^\infty\Omega_j$. Then for every $u\in\mathcal{F}(\Omega,f)$ such that*

$$\int_\Omega (dd^cu)^n<\infty,$$

there exists a decreasing sequence $u_j\in\mathcal{F}(\Omega_j,f)$ such that $u_j\searrow u$ in Ω and $(dd^cu_j)^n=(dd^cu)^n$ in Ω_j.

2.3 Approximation

As an application of the above result, in the next section of the chapter we study approximation of plurisubharmonic functions by an increasing sequence of plurisubharmonic functions defined on larger domains

Theorem 2.3.1. *Let Ω be a unbounded hyperconvex domain in $\mathbb{C}^n$ and let $\{\Omega_j\}_{j=1}^{\infty}$ be a sequence of unbounded hyperconvex domains such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$ and $PSH^s(\Omega_1) \cap L^{\infty}(\Omega_1) \neq \varnothing$. Assume that there exist $\psi \in \mathcal{F}(\Omega)$ and $\psi_j \in \mathcal{F}(\Omega_j)$ such that $\psi_j \nearrow \psi$ a.e in Ω as $j \nearrow \infty$. Then for every $f \in MPSH^-(\Omega_1) \cap \mathcal{C}(\Omega_1)$ and for every $u \in \mathcal{F}(\Omega, f)$ such that*

$$\int_{\Omega} (dd^c u)^n < \infty,$$

there exists $u_j \in \mathcal{F}(\Omega_j, f)$ such that $u_j \nearrow u$ a.e. in Ω as $j \nearrow \infty$.

we need the Proposition:

Proposition 2.3.2. *Let Ω be a unbounded hyperconvex domain in $\mathbb{C}^n$ such that $PSH^s(\Omega) \cap L^{\infty}(\Omega) \neq \varnothing$ and let $f \in MPSH^-(\Omega) \cap \mathcal{C}(\Omega)$. Assume that $u \in \mathcal{E}(\Omega, f)$ such that $\int\limits_{\Omega} (dd^c u)^n < \infty$. Then $u \in \mathcal{F}(\Omega, f)$ if and only if there exists a sequence $\{u_j\}_{j=1}^{\infty} \subset \mathcal{E}_0(\Omega, f)$ such that $u_j \searrow u$ in Ω as $j \nearrow \infty$ and*

$$\sup_j \int\limits_{\Omega} (dd^c u_j)^n < \infty.$$

Chapter 3

Subextension of m-subharmonic functions

As in overview,in this chapter,we do researchon on subextension of m - subharmonic in $\mathcal{F}_m(\Omega)$ with Ω being m - hyperconvex bounded in $\mathbb{C}^n$. We also indicate that an equality of the complex Hessian measures of subextension and initial function.

Chapter 3 includes two parts. In the first part, we will present background knowledge for this chapter. In the second one, we will prove several clauses, lemmas applied to prove the result of subextension of m - subharmonic functions and its consequence.

Chapter 3 was pulled out from the article [4] (in the mentioned project category of the thesis).

3.1 Some definitions and consequences

Let Ω be an open subset in $\mathbb{C}^n$ with the canonical Kähler form $\beta = dd^c\|z\|^2$ where $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{4}(\overline{\partial} - \partial)$ and, hence, $dd^c = \frac{i}{2}\partial\overline{\partial}$. For $1 \leq m \leq n$, following Bloki, we define

$$\Gamma_m = \{\eta \in C_{(1,1)} : \eta \wedge \beta^{n-1} \geq 0, \ldots, \eta^m \wedge \beta^{n-m} \geq 0\},$$

where $C_{(1,1)}$ denotes the space of $(1,1)$-forms with constant coefficients.

Definition 3.1.1. *Let u be a subharmonic function on an open subset $\Omega \subset \mathbb{C}^n$. u is said to be m-subharmonic function on Ω if for every $\eta_1, \ldots, \eta_{m-1}$ in Γ_m the inequality*

$$dd^c u \wedge \eta_1 \wedge \ldots \wedge \eta_{m-1} \wedge \beta^{n-m} \geq 0,$$

holds in the sense of currents.

By $SH_m(\Omega)$ we denote the set of m-subharmonic functions on Ω while $SH_m^-(\Omega)$ denotes the set of negative m-subharmonic functions on Ω.
Before to formulate basic properties of m-subharmonic functions, we recall the following. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ define

$$S_m(\lambda) = \sum_{1 \leq j_1 < \cdots < j_m \leq n} \lambda_{j_1} \cdots \lambda_{j_m}.$$

By $\mathcal{H}$ we denote the vector space of complex hermitian $n \times n$ matrices over $\mathbb{R}$. For $A \in \mathcal{H}$ let $\lambda(A) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be the eigenvalues of A. Put

$$\widetilde{S}_m(A) = S_m(\lambda(A)).$$

As in L.Garding(1959), we define

$$\widetilde{\Gamma}_m = \{A \in \mathcal{H} : \lambda(A) \in \Gamma_m\} = \{\widetilde{S}_1 \geq 0\} \cap \cdots \cap \{\widetilde{S}_m \geq 0\}.$$

Now we list the basic properties of m-subharmonic functions the proofs of which are repeated as for plurisubharmonic functions in pluripotential theory so we omit.

Proposition 3.1.2. *Let Ω be an open set in $\mathbb{C}^n$. Then we have*

(a) $PSH(\Omega) = SH_n(\Omega) \subset SH_{n-1}(\Omega) \subset \cdots \subset SH_1(\Omega) = SH(\Omega)$. Hence, $u \in SH_m(\Omega)$, $1 \leq m \leq n$, then $u \in SH_r(\Omega)$, for every $1 \leq r \leq m$.

(b) If u is C^2 smooth then it is m-subharmonic if and only if the form $dd^c u$ is pointwise in $\widetilde{\Gamma}_m$.

(c) If $u, v \in SH_m(\Omega)$ and $\alpha, \beta > 0$ then $\alpha u + \beta v \in SH_m(\Omega)$.

(d) If $u, v \in SH_m(\Omega)$ then so is $\max\{u, v\}$.

(e) If $\{u_j\}_{j=1}^{\infty}$ is a family of m-subharmonic functions, $u = \sup_j u_j < +\infty$ and u is upper semicontinuous then u is a m-subharmonic function.

(f) If $\{u_j\}_{j=1}^{\infty}$ is a decreasing sequence of m-subharmonic functions then so is $u = \lim_{j \to +\infty} u_j$.

(g) Let $\rho \geq 0$ be a smooth radial function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \rho dV_n = 1$, where dV_n denotes the Lebesgue measure of $\mathbb{C}^n$. For $u \in SH_m(\Omega)$ we define

$$u_\varepsilon(z) := (u * \rho_\varepsilon)(z) = \int_{\mathbb{B}(0,\varepsilon)} u(z - \xi)\rho_\varepsilon(\xi) dV_n(\xi), \ \forall z \in \Omega_\varepsilon,$$

where $\rho_\varepsilon(z) := \frac{1}{\varepsilon^{2n}}\rho(z/\varepsilon)$ and $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$. Then $u_\varepsilon \in SH_m(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.

(h) Let $u_1, \ldots, u_p \in SH_m(\Omega)$ and $\chi : \mathbb{R}^p \to \mathbb{R}$ be a convex function which is non decreasing in each variable. If χ is extended by continuity to a function $[-\infty, +\infty)^p \to [-\infty, \infty)$, then $\chi(u_1, \ldots, u_p) \in SH_m(\Omega)$.

Example 3.1.3. Let $u(z_1, z_2, z_3) = 3|z_1|^2 + 2|z_2|^2 - |z_3|^2$. By using (b) of Proposition 3.1.2 it is easy to see that $u \in SH_2(\mathbb{C}^3)$. However, u is not a plurisubharmonic function in $\mathbb{C}^3$ because the restriction of u on the line $(0, 0, z_3)$ is not subharmonic.

Now, we define the complex Hessian operator of locally bounded m-subharmonic functions as follows.

Definition 3.1.4. *Assume that $u_1, \ldots, u_p \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$. Then the complex Hessian operator $H_m(u_1, \ldots, u_p)$ is defined inductively by*

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m} = dd^c(u_p dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m}).$$

From the definition of m-subharmonic functions and using arguments as in the proof of Theorem 2.1 in Bedford and Taylor(1982) we note that $H_m(u_1, \ldots, u_p)$ is a closed positive current of bidegree $(n-m+p, n-m+p)$ and this operator in continuous under decreasing sequences of locally bounded m-subharmonic functions. Hence, for $p = m$, $dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}$ is a nonnegative Borel measure. In particular, when $u = u_1 = \cdots = u_m \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ the Borel measure

$$H_m(u) = (dd^c u)^m \wedge \beta^{n-m},$$

is well defined and is called the complex Hessian of u.

Similarly as the concept 1.1.1 about subextension of plurisubharmonic function, we define subextension of m-subharmonic function,

Definition 3.1.5. *Let $\Omega \subset \widetilde{\Omega}$ be open subsets of $\mathbb{C}^n$ and u a m-subharmonic function on Ω ($u \in SH_m(\Omega)$). A function $\widetilde{u} \in SH_m(\widetilde{\Omega})$ is said to be subextension of u if for all $z \in \Omega, \widetilde{u}(z) \leq u(z)$.*

Now, We recall m-hyperconvex domains in $\mathbb{C}^n$ which are useful for theory of m-subharmonic functions and the complex Hessian operator and they are similar as hyperconvex domains in pluripotential theory.

Definition 3.1.6. *Let Ω be a bounded domain in $\mathbb{C}^n$. Ω is said to be m-hyperconvex if there exists a continuous m-subharmonic function $u : \Omega \longrightarrow \mathbb{R}^-$ such that $\Omega_c = \{u < c\} \Subset \Omega$ for every $c < 0$.*

Remark 3.1.7. From the definition of m-hyperconvex domains and the definition of m-subharmonic fuctions, we see that for all plurisubharmonic functions are m-subharmonic functions with all $n \geq m \geq 1$ so that all hyperconvex domains in $\mathbb{C}^n$ are m-hyperconvex domains.

Next, as in L.H.Chinh(2013, 2015) we recall Cegrell's classes for m-subharmonic functions as follows. .

Definition 3.1.8. *Let $\Omega \subset \mathbb{C}^n$ be a m-hyperconvex domain. Put:*

$$\mathcal{E}_m^0 = \mathcal{E}_m^0(\Omega) = \{u \in SH_m^-(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} u(z) = 0, \quad \int_\Omega H_m(u) < \infty\}.$$

$$\mathcal{F}_m = \mathcal{F}_m(\Omega) = \big\{u \in SH_m^-(\Omega) : \exists \ \ \mathcal{E}_m^0 \ni u_j \searrow u, \quad \sup_j \int_\Omega H_m(u_j) < \infty\big\}.$$

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \big\{u \in SH_m^-(\Omega) : \forall z_0 \in \Omega, \exists \ \ a\ neighborhood \ \ \omega \ni z_0, \quad v \in \mathcal{E}_m^0 \ni u_j \searrow u \ \ on \ \ \omega, \sup_j \int_\Omega H_m(u_j) < \infty\big\}.$$

Remark 3.1.9.
a) From the above definitions, it is easy to see that $\mathcal{E}_m^0(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{E}_m(\Omega)$.
b) Similar as Theorem 4.5 in Cegrell, Theorem 3.5 in L.H.Chinh implies that the class $\mathcal{E}_m$ is the biggest class of $SH_m^-(\Omega)$ satisfying the conditions
(i) if $u \in \mathcal{E}_m(\Omega)$ and $v \in SH_m^-(\Omega)$ then $\max\{u, v\} \in \mathcal{E}_m(\Omega)$.
(ii) if $u \in \mathcal{E}_m(\Omega)$ and $u_j \in SH_m^-(\Omega) \cap L^\infty_{loc}(\Omega), u_j \searrow u$, then $H_m(u_j)$ is weakly convergent.

Similar as in pluripotential theory ones defines the relative m-extremal functions as follows.

Definition 3.1.10. *Let Ω be an open subset of $\mathbb{C}^n$ and $E \subset \Omega$. The relative m-extremal function of the pair (E, Ω) is defined by*

$$h_{m,E,\Omega} = h_{m,E} = \sup\{u \in SH_m^-(\Omega) : u|_E \leq -1\}.$$

As L.H.Chinh(2015), $h^*_{m,E}$ is a negative m-subharmonic function in Ω. Moreover, if Ω is a m-hyperconvex domain in $\mathbb{C}^n$ and $\Omega' \Subset \Omega$ then it is easy to prove that $h_{m,\Omega'}$ belongs to $\mathcal{E}_m^0(\Omega)$.

Similar as in pluripotential theory ones defines m-polar subsets and Josefson's theorem for m-polar subsets

Definition 3.1.11. *Let Ω be an open subset in $\mathbb{C}^n$ and $E \subset \Omega$. E is said to be m-polar if for any $z \in E$ there exists a connected neighbourhood V of z in Ω and $v \in SH_m(V), v \not\equiv -\infty$ such that $E \cap V \subset \{v = -\infty\}$.*

Theorem 2.35 in L.H.Chinh(2013) shows that the Josefson theorem in pluripotential theory is valid for m-polar sets. That means that if $E \subset \Omega$ is an m-polar set then there exists an m-subharmonic function in $\mathbb{C}^n$ such that $E \subset \{u = -\infty\}$ on E.

Remark 3.1.12.
a) By (a) of Proposition 3.1.2 it follows that every pluripolar set in pluripotential theory is m-polar for all $1 \leq m \leq n$.
b) By Example 2.27 in L.H.Chinh(2015) we note that there exists an m-polar set E^n in $\mathbb{C}^n$ which is not a pluripolar set.

3.2 Subextension in class $\mathcal{F}_m(\Omega)$

In this section we will present the results about subextensions in class $\mathcal{F}_m(\Omega)$. We will prove the theorem

Theorem 3.2.1. *Let $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}^n$ be bounded m-hyperconvex domains and $u \in \mathcal{F}_m(\Omega)$. Then there exists $w \in \mathcal{F}_m(\widetilde{\Omega})$ such that $w \leq u$ on Ω and*

$$(dd^c w)^m \wedge \beta^{n-m} = 1_\Omega (dd^c u)^m \wedge \beta^{n-m}.$$

First we need the following proposition which is similar as Lemma 2.1 of Cegrell - Kolodziej and Zeriahi and is used in the proof of subextension for m-subharmonic functions in the class $\mathcal{F}_m$.

Proposition 3.2.2. *Let Ω is an m-hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}_m(\Omega)$ then*

$$e_m(u) = \int_\Omega H_m(u) < \infty.$$

We have to use the result.

Proposition 3.2.3. *Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$ and $\{u_j\} \subset \mathcal{F}_m(\Omega)$ be a decreasing sequence which converges to $u \in \mathcal{F}_m(\Omega)$. If $\varphi \in SH_m^-(\Omega) \cap L^\infty(\Omega)$ then*

$$\lim_j \int_\Omega \varphi H_m(u_j) = \int_\Omega \varphi H_m(u).$$

Next, we need the following lemma which is used in the proof of Theorem 3.2.1. It also gives a new technique in the approach to subextension of m-subharmonic functions with the control of complex Hessian measures.

Lemma 3.2.4. *Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}_m(\Omega)$. Then there exist $g \in \mathcal{F}_m^a(\Omega), h \in \mathcal{F}_m(\Omega)$ such that*

$$1_{\{u>-\infty\}}(dd^c u)^m \wedge \beta^{n-m} = (dd^c g)^m \wedge \beta^{n-m}, \tag{3.1}$$

$$1_{\{u=-\infty\}}(dd^c u)^m \wedge \beta^{n-m} = (dd^c h)^m \wedge \beta^{n-m} \tag{3.2}$$

and $h \geq u \geq g + h$ on Ω.

From the above theorem, we obtain the following corollary.

Corollary 3.2.5. *Let $\Omega \subset \widetilde{\Omega}$ be bounded m-hyperconvex domains and $\{u_j\}_{j\geq 1}, u \subset \mathcal{F}_m(\Omega)$ be such that $u_j \geq u$, u_j is convergent in C_m-capacity to u on Ω. Assume that $\widetilde{u}_j, \widetilde{u}$ are subextensions of u_j, u, respectively, to $\widetilde{\Omega}$. Then $H_m(\widetilde{u}_j)$ is weakly convergent to $H_m(\widetilde{u})$ on $\widetilde{\Omega}$.*

Chapter 4

Equations of complex Monge-Ampère type for arbitrary measures

As in the introduction part. The purpose of this project is to present the existence of weak solutions of equations of complex Monge Ampre type for arbitrary, in particular, measures carried by pluripolar sets.

Chapter 4 includes two parts. In the first part, we will introduce about equations of complex Monge Ampre type and the demonstration of the main result of the chapter. In the second one, we will prove the existence of weak solutions of complex Monge Ampre type on $\mathcal{N}(\Omega, f)$ class for arbitrary measures.

Chapter 4 was based on the article [3] (in the mentioned project category of the thesis).

4.1 Introduction

To be suitable for the presentation, we will recall the definition of equations of complex Monge Ampre type released by Bedford, Taylor.

Definition 4.1.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and μ a positive Borel measure on Ω. Assume that $F : \mathbb{R} \times \Omega \longrightarrow [0, +\infty)$ is a $dt \times d\mu$-measurable function. The equation of the form*

$$(dd^c u)^n = F(u, .)d\mu, \tag{4.1}$$

where u is a plurisubharmonic function on Ω is called to be the equation of complex Monge-Ampère type

Bedford and Taylor proved the existence of a solution to the following Monge-Ampre type equation (4.1). They assumed that μ is the Lebesgue measure, and $F^{\frac{1}{n}} \geq 0$ is bounded, continuous, convex, and increasing in the first variable. Late in 1984, Cegrell showed that the convexity and monotonicity conditions are superfluous. The case when F is smooth was proved. Kołodziej proved existtence and uniqueness of soluion to (4.1) when F is a bounded, nonnegative function that is nondecreasing and continuous in the first variable. Furthermore, μ was assumed to be a Monge - Ampère measure generaed by some bounded plurisubharmonic function and Ω is strictly pseudoconvex. A generalization to hyperconvex domains was made by Cegrell and Kołodziej. There assumption were that $\mu(\Omega) < +\infty$ and μ vanishing on pluripolar sets, if $0 \leq F(t, z) \leq g(z)$ with $g \in L^1(d\mu)$ then for all $f \in MPSH(\Omega) \cap \mathcal{E}(\Omega)$,Cegrell and Kołodziej proved that equation (4.1) has a solution $u \in \mathcal{F}^a(\Omega, f)$.

Late, Czyż investigated the equation (4.1) in the class $\mathcal{N}(\Omega, f)$. He proved that if μ vanishes on pluripolar sets of Ω, F is a continuous function of the first variable and bounded by an integrable function for $(-\varphi)\mu$ which is independent of the first variable then the equation (1.1) is solvable in the class $\mathcal{N}(\Omega, f)$. More recently, under the same assumption that μ vanishes on all pluripolar sets of Ω and there exists a subsolution $v_0 \in \mathcal{N}^a(\Omega)$, i.e there exists a function $v_0 \in \mathcal{N}^a(\Omega)$ such that $(dd^c v_0)^n \geq F(v_0, .)d\mu$, Benelkourchi showed that (4.1) has a solution $u \in \mathcal{N}^a(\Omega, f)$.

In this note we want to study weak solutions of the equation (4.1) on class $\mathcal{N}(\Omega, f)$ for an arbitrary measure, in particular, for measures carried by a pluripolar set.
When solving the problems above, we had difficulties when μ is carried by a pluripolar set then hot to solve the problems. To solve the problems, firstly, we find weak solutions for measures carried by a pluripolar. Then we build a boundary type Perron - Bremerman plurisubharmonic functions different from other authors to continue solving other parts. To be in more details, we will now prove the main result of the chapter.

4.2 Equations of complex Monge-Ampère type for arbitrary measures

We achieve the result:

Theorem 4.2.1. *Let Ω be a bounded hyperconvex domain and μ be a nonnegative measure in Ω. Assume that $F : \mathbb{R} \times \Omega \longrightarrow (0, +\infty)$ is a $dt \times d\mu$-measurable function such that:*

(1) For all $z \in \Omega$, the function $t \longmapsto F(t, z)$ is continuous and nondecreasing.

(2) For all $t \in \mathbb{R}$, the function $z \longmapsto F(t, z)$ belongs to $L^1_{loc}(\Omega, \mu)$.

(3) There exists a function $w \in \mathcal{N}(\Omega)$ such that $(dd^c w)^n \geq F(w, .)d\mu$.

Then for any maximal plurisubharmonic function $f \in \mathcal{E}(\Omega)$ there exists $u \in \mathcal{N}(\Omega, f)$ such that $u \geq w$ and $(dd^c u)^n = F(u, .)d\mu$ in Ω.

We need the following.

Lemma 4.2.2. *Let Ω, μ, F and w satisfy all the hypotheses of Theorem 4.2.1. Assume that $w \in \mathcal{N}^a(\Omega)$, $suppd\mu \Subset \Omega$, $d\mu(\Omega) < \infty$ and $d\mu$ vanishes on all pluripolar sets of Ω. If $f \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$ and $v \in \mathcal{F}(\Omega, f)$ such that $supp(dd^c v)^n \Subset \Omega$ and $(dd^c v)^n$ is carried by a pluripolar set of Ω, the function*

$$u := (\sup\{\varphi \in \mathcal{E}(\Omega) : \varphi \leq v \text{ and } (dd^c\varphi)^n \geq F(\varphi, .)d\mu\})^*.$$

belongs to $\mathcal{N}(\Omega, f)$ and $(dd^c u)^n = F(u, .)d\mu + (dd^c v)^n$ in Ω.

Conclusions and recommendations

I.Conclusions

The thesis has attained the proposed research purposes. Its results help enrich subextension of unbounded plurissubharmonic function in the class $\mathcal{E}_\chi(\Omega, f)$, $\mathcal{F}(\Omega, f)$, $\mathcal{F}_m(\Omega)$ with the control over the weighted Monge - Ampère measure and the complex Hessian measure.

1) Successfully proved the existence of subextension in the class $\mathcal{E}_\chi(\Omega, f)$ in the case Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and as well as indicated the equality $\chi(\widetilde{u})(dd^c\widetilde{u})^n = 1_\Omega\chi(u)(dd^c u)^n$ on $\widetilde{\Omega}$.

2) Solved the subextension problem with answer for the class $\mathcal{F}(\Omega, f)$ in the case Ω is an unbounded hyperconvex domain in $\mathbb{C}^n$ and denoted the equality of the weighted Monge - Ampère mesure of subextension and of the given function.

3) Extended Hed's result for approximattion of plurissubharmonic functions by an increasing sequence of plurissubharmonic functions defined on larger domains in the class $\mathcal{F}(\Omega, f)$ in the case Ω is an unbounded hyperconvex domain in $\mathbb{C}^n$.

4) Proved the existence of subextension and the equality among complex Hessian measures for the class $\mathcal{F}_m(\Omega)$ in m - subharmonic functions.

5) Established the existence of weak solutions belonging to the class $\mathcal{N}(\Omega, f)$ of equations of complex Monge - Ampère type for arbitrary measures.

II. Recommendations

We suggest that in the near future, finding Holder continuous solutions for equations of complex Monge - Ampère type to the complex Monge - Ampère and Hessian operator be one of problems of interest and in need of being solved. We specially have to investigate this problem for other larger subjects as compared to domains in $\mathbb{C}^n$, such as those on the Kahler compact variety or more generally, on Hermite varieties. There have been several achievements attained by this direction for the time being, however, a complete answer for this direction of investigation is expected to be far from reaching.

Papers used in the thesis

A. Papers used in the thesis.

[1] Le Mau Hai, Nguyen Xuan Hong and Trieu Van Dung (2015), "Subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes", *Complex Var. Elliptic Equ.*, 60(11), pp. 1580-1593.(SCIE)

[2] Le Mau Hai, Nguyen Van Khiem and Trieu Van Dung (2016), "Subextension of plurisubharmonic functions in unbounded hyperconvex domains and applications", *Complex Var. Elliptic Equ.*, 61(8), 1116–1132.(SCIE)

[3] Le Mau Hai, Tang Van Long and Trieu Van Dung (2016), "Equations of complex Monge-Ampère type for arbitrary measures and applications", *Int. J. Math.*, 27(4), 1650035(13 pages).(SCI)

[4] Le Mau Hai, Trieu Van Dung (2018), "Subextension of m-subharmonic functions", submitted to Vietnam J.Math.

B. Some reports of results of the thesis in conferences, seminars.

[1] Trieu Van Dung (2014), *"Subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes"* , Report of Scientific Conference Faculty of Mathematics - Informations HNUE.

[2] Trieu Van Dung (2016), *"Subextension of plurisubharmonic functions in unbounded hyperconvex domains and applications"*, Report of Scientific Conference Faculty of Mathematics - Informations HNUE..

[3] Trieu Van Dung (2018), *"Equations of complex Monge-Ampère type for arbitrary measures and applications"* , Report of Scientific Conference Faculty of Mathematics - Informations HNUE.

[4] Trieu Van Dung (2018), "Subextension of m-subharmonic functions", August 2018 Report to the 9th National Mathematics Congress in Nha Trang.